UNIT 7: APPLICATIONS OF VECTORS

7.1 VECTORS AS FORCES

- Examples of force: muscular exertion, gravitational pull, magnetic attraction, engine thrust, shock absorbers.
- Force can be defined as that which changes, or tends to change, the state of rest, or uniform motion of a body.
- Force is a **vector quantity** because it has a magnitude and a direction.
- On Earth, force is defined as the product between the mass of an object and the acceleration due to gravity ($9.8 m/s^2$). So a 1kg mass exerts a downward force of $1kg \times 9.8 m/s^2$ or $9.8 kg \bullet m/s^2$. This unit of measure is called a Newton and is abbreviated as N.
- The **resultant** of several forces is the single force that can be used to represent the combined effect of all the forces. The individual forces that make up the resultant are referred to as the components of the resultant.
- The **equilibrant** of a number of forces is the single force that opposes the resultant of the forces acting upon an object. When the equilibrant is applied to the object, this force maintains the object in a state of equilibrium.
- In order to find the resultant and equilibrant of forces, we can use the properties of vector addition. To find a resultant vector, use either the parallelogram or triangle rule to find the sum of $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$. For vectors in equilibrium, placing them head to tail will result in a triangle, because the resultant of two of the forces will be opposed by the third.

Ex1. Two children, James and Fred, are pushing on a rock. James pushes with a force of 80N in an easterly direction, and Fred pushes with a force of 60N in the same direction. Determine the resultant and the equilibrant of these two forces.

Ex2. Two forces of 20N and 40N act at an angle of 30° to each other. Determine the resultant of these two forces, the equilibrant and their relative positions.

Ex3. Kayla pulls on a rope attached to her sled with a force of 200N. If the rope makes an angle of 20° with the horizontal, determine

a) The force that pulls the sled forward
b) The force that tends to lift the sled

Ex4. A mass of 20kg is suspended from a ceiling by two lengths of rope that make angles of 60° and 45° with the ceiling. Determine the tension in each of these ropes.

Ex5. A) is it possible for three forces of 15N, 18N and 38N to keep a system in a state of equilibrium? (NB the triangle inequality states that for a triangle to be formed, the sum of any two sides must be greater than or equal to the third side.)

b) Three forces have a magnitude of 3N, 5N and 7N, and are in a state of equilibrium. Calculate the angle between the two smaller forces when they are placed tail to tail.

7.2 VELOCITY

We can apply what we have learned about vectors to questions involving velocity.

1) When calculating resultant velocities, it is necessary to draw a triangle showing the velocities (a vector diagram) so that you may see the relationship between the two component vectors and their resultant.
2) The velocity of an object is stated relative to a frame of reference. The frame of reference used influences the stated velocity of the object. For example, an airplane's velocity is understood to be its airspeed (the speed relative to a person on board the craft). When influenced by a wind (whose velocity is stated relative to the ground, a fixed point), the resultant vector is the velocity of the airplane relative to the ground, and is called the ground velocity. Pay careful attention to these details.

Ex1. A plane is headed due north with an air speed of 400km/h when it is blown off course by a wind of 100km/h from the northeast. Determine the resultant ground velocity of the airplane.

Ex2. A river is 2km wide and flows at 6km/h. Anna is driving a motorboat, which has a speed of 20km/h in still water and she heads out from one bank in a direction perpendicular to the current. A marina lies directly across the river from the starting point on the opposite bank.

a) How far downstream from the marina will the current push the boat?
b) How long will it take for the boat to cross the river?
c) If Anna decides that she wants to end up directly across the river at the marina, in what direction should she head? What is the resultant velocity of the boat?

Ex3. A child, sitting in the backseat of a car travelling 20m/s, throws a ball at 2m/s to her brother who is sitting in the front seat.

a) What is the velocity of the ball relative to the children?
b) What is the velocity of the ball relative to the road?

7.3 THE DOT PRODUCT OF TWO GEOMETRIC VECTORS

The dot product between two geometric vectors is a scalar quantity defined as:

$\vec{a}\bullet\vec{b} = |\vec{a}||\vec{b}|cos\theta$, where θ is the angle between the two vectors. The dot product has the primary application of telling us the angle between two vectors. There are a few necessary properties associated with the dot product:

1) Commutative Property: $\vec{a}\bullet\vec{b} = \vec{b}\bullet\vec{a}$ (because the dot product deals with scalar properties)
2) Distributive Property: $\vec{p}\bullet(\vec{q}+\vec{r}) = \vec{p}\bullet\vec{q}+\vec{p}\bullet\vec{r}$
3) Magnitudes Property: $\vec{p}\bullet\vec{p} = |\vec{p}|^2$ (because cos 0 ° = 1)
4) Associative Property with a scalar k: $(k\vec{p})\bullet\vec{q} = \vec{p}\bullet(k\vec{q}) = k(\vec{p}\bullet\vec{q})$

Consider too that the angle θ will tell you whether the dot product is to be negative, positive, or 0.

For cos θ to be positive, $0^\circ \le \theta < 90^\circ$. The dot product will be positive.

Cos 90° = 0, so the dot product will also be 0. *Thus, two non-zero perpendicular vectors will always have a dot product of 0 – this will be important to us later on in the course.*

For cos θ to be negative, $90^\circ < \theta \le 180^\circ$. The dot product will also be negative.

We only calculate the dot product for angles between 0 and 180° inclusive.

Ex. Two vectors $\vec{a}$ and $\vec{b}$ are placed tail to tail and have magnitudes 3 and 5. Calculate $\vec{a}\bullet\vec{b}$ if the angle between the two vectors is 120°

Ex. If the vectors $\vec{a}+3\vec{b}$ and $4\vec{a}-\vec{b}$ are perpendicular, and $|\vec{a}| = 2|\vec{b}|$, determine the angle to the nearest degree between the non-zero vectors $\vec{a}$ and $\vec{b}$.

Ex. If $|\vec{x}+\vec{y}| = |\vec{x}-\vec{y}|$, prove that the non-zero vectors $\vec{x}$ and $\vec{y}$ are perpendicular.

7.4 THE DOT PRODUCT OF ALGEBRAIC VECTORS

Recall:

1) $\vec{a} = (a_1, a_2, a_3)$ is an algebraic vector.
2) $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$
3) $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

Using this knowledge, we will set out to prove a theorem: in R^3, $\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = \vec{b} \bullet \vec{a}$. (We
a● b=a1b1+a2b2

Notes:

1) The same properties hold for geometric and algebraic vectors.
2) For two non-zero vectors, $\cos\theta = \frac{\vec{a} \bullet \vec{b}}{|\vec{a}||\vec{b}|}$

Ex. Given the vectors (-1, 2, 4) and (3, 4, 3), calculate the dot product and the angle between the two vectors.

Ex. For what values are the following pairs of vectors perpendicular?

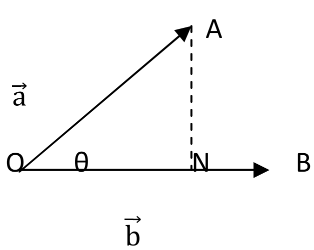
a) (-1, 3, -4) and (3, k, -2)
b) (m, m, -3) and (m, -3, 6)

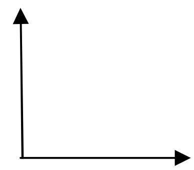
Ex. A parallelogram has sides defined by $\vec{a} = (6, 1)$ and $\vec{b} = (-1, 3)$. Determine the angle between the diagonals of the parallelogram formed by these vectors.

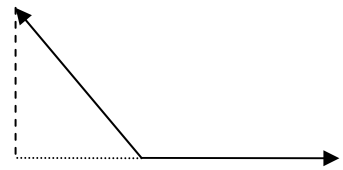
Ex. Find a vector (or vectors) perpendicular to each of the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$.

7.5 SCALAR AND VECTOR PROJECTIONS

Def. Given two vectors, $\vec{a}$ and $\vec{b}$, placed tail to tail, and with an angle between them of $0^\circ \leq \theta \leq 180^\circ$, the scalar projection is found by drawing a line from the head of $\vec{a}$ perpendicular to $\vec{b}$, as shown. The projection made is a scalar value, and is obtained by using right-triangle trigonometry (SOHCAHTOA).



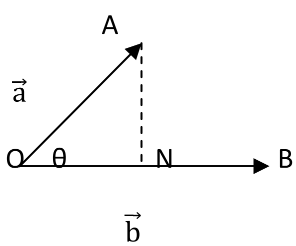




A couple of primary observations:

1) The sign of the scalar projection corresponds to what we know about the dot product.
2) The projection of $\vec{a}$ on $\vec{b}$ is generally not the same as the projection of $\vec{b}$ on $\vec{a}$.
3) You cannot take the scalar projection of $\vec{a}$ onto $\vec{0}$.
4) The scalar projection of $\vec{a}$ on $\vec{b}$ is independent of the length of $\vec{b}$.

To take the scalar projection, let's consider the acute angle case seen above.



If we look at the right triangle formed, we are trying to solve for ON. Since we know the angle θ and $|\vec{a}|$, we can use the cosine ratio: $\cos\theta = \frac{ON}{|\vec{a}|}$, so ON = $|\vec{a}|\cos\theta$.

Connecting this to what we know of the dot product:

$$\vec{a} \bullet \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\vec{a} \bullet \vec{b} = |\vec{a}|\cos\theta\,|\vec{b}|$$

$|\vec{a}|\cos\theta = \frac{\vec{a}\bullet\vec{b}}{|\vec{b}|}$. This is the scalar projection of $\vec{a}$ on $\vec{b}$. The scalar projection of $\vec{b}$ on $\vec{a}$ is $\frac{\vec{a}\bullet\vec{b}}{|\vec{a}|}$.

Ex. Given $\vec{a} = \left(-3, 4, 5\sqrt{3}\right)$ and $\vec{b} = (-2, 2, 1)$, determine the scalar projection of $\vec{a}$ on $\vec{b}$ and $\vec{b}$ on $\vec{a}$.

Ex. Given the vector $\overrightarrow{OP} = (2, 1, 4)$, determine the angle that it makes with each of the coordinate axes.

This previous example is important because for any vector $\overrightarrow{OP} = (a, b, c)$, we can determine the angle between it and each coordinate axis. The cosines of the angles are called the **direction cosines of** $\boldsymbol{\alpha, \beta}$ **and** $\boldsymbol{\gamma}$.

x-axis: cos α = $\frac{\boldsymbol{a}}{\sqrt{\boldsymbol{a^2+b^2+c^2}}}$ **y-axis: cos β =** $\frac{\boldsymbol{b}}{\sqrt{\boldsymbol{a^2+b^2+c^2}}}$ **z-axis: cos γ =** $\frac{\boldsymbol{c}}{\sqrt{\boldsymbol{a^2+b^2+c^2}}}$

Vector Projections

We can connect what we know about scalar projections to vector projections, by taking our formula for scalar projection, $\frac{\vec{a}\bullet\vec{b}}{|\vec{b}|}$, and multiplying it by $\frac{\vec{b}}{|\vec{b}|}$, which is a unit vector pointing in the direction of b.

Ex. Find the vector projection of $\overrightarrow{OA} = (4, 3)$ on $\overrightarrow{OB} = (4, -1)$.

7.6 THE CROSS PRODUCT OF TWO VECTORS

The cross product of two vectors, $\vec{a}$ and $\vec{b}$, is a third vector that is perpendicular to both (in 3D space). An infinite number of vectors satisfy this condition, all of which are scalar multiples of each other, so we use the simplest one in our answer.

$\vec{a} \times \vec{b}$ is not the same vector as $\vec{b} \times \vec{a}$, so we must have a way of differentiating them. We shall use the right-handed system that we used at the beginning of the vectors unit. By using your right hand, and placing your index finger on $\vec{a}$ and your middle finger on $\vec{b}$, (thus having to rotate $\vec{a}$ counterclockwise to become collinear with $\vec{b}$), you may see that your thumb gives you a vector perpendicular to $\vec{a}$ and b.

The vector $\vec{b} \times \vec{a}$ is the opposite, and is perpendicular to $\vec{a}$ and $\vec{b}$. Using your hand, this vector is formed when $\vec{a}$ is rotated clockwise onto $\vec{b}$.

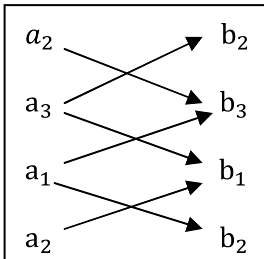
The formula for the cross product is a bit difficult to remember, but we will employ a trick.

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

$$\vec{b} \times \vec{a} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)$$

In order to calculate $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$:

1) List the components of vector $\vec{a}$ in column form on the left side, starting with a_2 and then writing a_3, a_1, a_2.
2) Write the components of $\vec{b}$ in a similar way on the right hand side.
3) Follow the arrows to find the x, y and z components of the cross product.



x = $a_2b_3 - a_3b_2$

y = $a_3b_1 - a_1b_3$

z = $a_1b_2 - a_2b_1$

Ex. Calcuate $\vec{d} \times \vec{e}$ and $\vec{e} \times \vec{d}$ if $\vec{d} = (-1, 3, 2)$ and $\vec{e} = (2, -5, 6)$. Check that your answers are perpendicular to the given vectors using the dot product.

Properties of the Cross Product:

1) Vector multiplication is not commutative: $\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$
2) Distributive law for vector multiplication: $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$
3) Scalar Law for vector multiplication: $k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$

7.7 APPLICATIONS OF THE DOT PRODUCT AND CROSS PRODUCT

Physical Application of the Dot Product

Def. WORK is defined as when a force, acting on an object, moves that object from one point to another. The formula for the calculation of work is W = $\vec{f} \bullet \vec{s}$, where $\vec{f}$ is the force acting on an object, measured in newtons (N), $\vec{s}$ is the displacement of th object, measured in metres (m), and W is the work done, measured in joules (J).

Ex. Marianna is pulling her daughter in a toboggan and is exerting a force of 40N, acting at 24° to the ground. If Marianna pulls the child a distance of 100m, how much work was done?

Geometric Application of the Cross Product

The cross product can be used to calculate the area of a parallelogram whose sides are defined by $\vec{a}$ and $\vec{b}$. The area of the parallelogram ABCD is $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.

Ex. Determine the area of the parallelogram determined by the vectors $\vec{p} = (-1,5,6)$ and $\vec{q} = (2,3,-1)$.

Ex. Determine the area of the triangle formed by the points A(-1,2, 1), B(-1, 0, 0) and C(3, -1, 4).

Ex. Without calculating, explain why $\vec{j} \times \vec{k} = \vec{i}$.

Physical Application of the Cross Product

We can also use the cross product in situations involving rotation. Example of this is the tightening or loosening of a nut using a wrench, opening a door, or pedaling a bicycle.

In the following example, a bolt with a right-hand thread is being screwed into a piece of wood using a wrench. A force $\vec{f}$ is being applied to the wrench at point N and is rotating about the point M. the vector $\vec{r}$ = $\overrightarrow{MN}$ is the position vector of N with respect to M.

The tourque, or turning effect, of the force $\vec{f}$ about the point M is defined to be the vector $\vec{r} \times \vec{f}$. Tourque can also be expressed as $|\vec{r}||\vec{f}|\sin\theta$. These two expressions are equal.

Ex. A 20N force is applied at the end of a wrench that is 40cm in length. The force is applied at an angle of 60° to the wrench. Calculate the magnitude of the tourque about the point of rotation M.